

I. Definition of a real numbers and Common sets of numbers

In mathematics, a real number is a number that can be used to measure a continuous one-dimensional quantity such as a distance, duration or temperature.

We consider the following subsets of $\mathbb{R}$:

- ▶ **Natural numbers** are denoted by $\mathbb{N}$.
- ▶ **Integers** are denoted by $\mathbb{Z}$.
- ▶ **Decimal numbers** are expressed in the decimal form $\frac{a}{10^n}$ where $a \in \mathbb{Z}, n \in \mathbb{N}$. The set of decimal numbers is denoted by D .
- ▶ **Rational numbers** can be expressed as the quotient or fraction $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$. The set of rational numbers is usually denoted by $\mathbb{Q}$.
- ▶ **Irrational numbers** are all numbers that are not rational, i. e, any real number that cannot be expressed as the quotient of two integers $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$. For example : $\sqrt{2}, \sqrt{3}...$, π and e are all irrational numbers are transcendental.

II. Fundamental Properties of real numbers set

II.1 Absolute Value

The absolute value of a real number x , denoted by $|x|$, is defined as :

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Note : Let A be a real-valued function : $|A(x)| = \begin{cases} A(x), & \text{if } A(x) \geq 0 \\ -A(x), & \text{if } A(x) < 0 \end{cases}$

Properties : The following properties hold :

1. $\forall x \in \mathbb{R}, |-x| = |x|$.
2. $\forall x \in \mathbb{R}, |x| = 0 \Leftrightarrow x = 0$.
3. $\forall x \in \mathbb{R}, \sqrt{x^2} = |x|$.
4. $\forall (x, y) \in \mathbb{R}^2, |x \times y| = |x| \times |y|$.
5. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}^*, \left| \frac{x}{y} \right| = \frac{|x|}{|y|}$.
6. $\forall x \in \mathbb{R}, \forall n \in \mathbb{N} |x^n| = |x|^n$.
7. $\forall (x, y) \in \mathbb{R}^2, |x + y| \leq |x| + |y|$ (Triangle Inequality).
8. $\forall x \in \mathbb{R}, \forall a \geq 0, |x| \leq a \Leftrightarrow -a \leq x \leq a$.
9. $\forall x \in \mathbb{R}, \forall a \in \mathbb{R}, |x| \geq a \Leftrightarrow x \geq a \text{ or } x \leq -a$.
10. $\forall (x, y) \in \mathbb{R}^2, |x| = |y| \Leftrightarrow x = y \text{ or } x = -y$.

II.2 Floor Function or integer part function

Definition : The **floor** of a real number x , denoted as $\text{floor}(x)$ (or $\lfloor x \rfloor$) is the greatest integer that is less than or equal to x . In other words, for any real number x , there exists a unique integer $\text{floor}(x) \in \mathbb{Z}$, such that

$$\text{floor}(x) \leq x < \text{floor}(x) + 1.$$

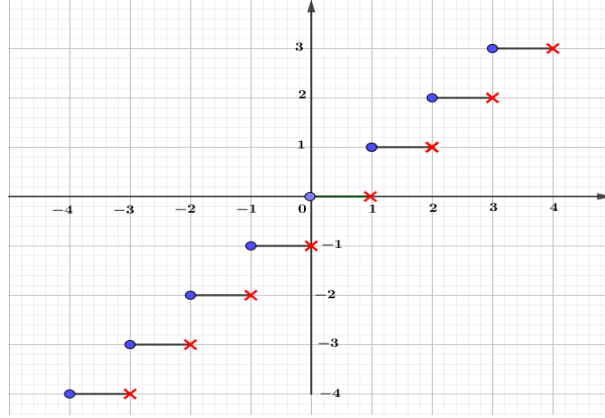


FIGURE 1 – Representative curve of the integer part function

Properties : Let x be a real number. We have

1. $x - 1 < \lfloor x \rfloor \leq x$.
2. $\lfloor x \rfloor = x \iff x \in \mathbb{Z}$.
3. $\forall k \in \mathbb{Z}, \lfloor x + k \rfloor = \lfloor x \rfloor + k$.

Examples : $\lfloor 3 \rfloor = 3$, $\lfloor -5 \rfloor = -5$, $\lfloor 2.8 \rfloor = 2$, $\lfloor -3.1 \rfloor = -4$, $\left\lfloor \frac{7}{11} \right\rfloor = 0$.

III. Bounds of subsets of $\mathbb{R}$

III.1 Maximum and minimum :

Definition : Let $A \subset \mathbb{R}$ be a nonempty subsets of $\mathbb{R}$.

► We say that α is the greatest element of A (or **maximum** of A) if and only if

$$\alpha \in A \quad \text{and} \quad \forall x \in A, \quad x \leq \alpha.$$

We denote $\alpha = \max A$.

► We say that β is the least element of A (or **minimum** of A) if and only if

$$\beta \in A \quad \text{and} \quad \forall x \in A, \quad x \geq \beta.$$

Notice that $\beta = \min A$.

Examples :

1. The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ do not have a greatest element.
2. The set $\mathbb{N}$ has a least element $0 \in \mathbb{N}$ ($\min \mathbb{N} = 0$ because $\forall x \in \mathbb{N}, x \geq 0$).
3. $A = [2, 6]$ has a maximum and minimum, $\min A = 2$ and $\max A = 6$.
4. $B =]1, 7[$ has neither maximum nor minimum.
5. $C =]-2, 3]$ has a maximum and does not have a minimum, $\max C = 3$ and the minimum C does not exist

III.2 Upper and lower bounds (*majorant and minorant*) :

Definition : Let m and M be two arbitrary real numbers, and let A be a nonempty subset of $\mathbb{R}$.

► We say that M is an upper bound of the set A (or A is bounded above by M) if and only if

$$\forall x \in A, x \leq M.$$

► We say that m is a lower bound of the set A (or A is bounded below by m) if and only if

$$\forall x \in A, x \geq m.$$

► If the set A is both bounded above and bounded below, then A is said to be bounded.

Remark :

- "The least element of A is a lower bound of A , and the greatest element of A is an upper bound of A ."
- "A lower bound of A (respectively, an upper bound of A) may not be the least element of A (respectively, the greatest element of A), as it is not necessarily in A ."
- "The least element and the greatest element, if they exist, are unique."

Examples :

1. $A = [2, 6]$ is bounded ; it is both bounded above and bounded below. The set of upper bounds of A is $[6, +\infty[$ and the set of lower bounds of A is $] - \infty, 2]$.
Hence, we have $\max A = 6$ and $\min A = 2$.
2. $B =]1, 7[$ is bounded ; it is both bounded above and bounded below.
The set of upper bounds of B is $[7, +\infty[$ and the set of lower bounds of B is $] - \infty, 1]$.
Thus, 7 is an upper bound of B but it is not a maximum of B , and we write $\max B$ does not exist.
1 is a lower bound of B but it is not a minimum of B , and we write $\min B$ does not exist.
3. $C =] - \infty, 5]$ is bounded above, and the set of upper bounds of C is $[5, +\infty[$, but it is not bounded because it is not bounded below.

III.3 Supremum and infimum (sup and inf)

Definition : Let A be a nonempty subset of $\mathbb{R}$.

- The supremum of A , denoted as $\sup A$; is the least upper bound of A .
- The infimum of A , denoted as $\inf A$; is the greatest lower bound of A .

Examples :

1. $A = [2, 6]$ the set of upper bounds of A is $[6, +\infty[$, then, $\sup A = 6$.
The set of lower bounds of A is $] - \infty, 2]$, then, $\inf A = 2$.
2. $B =]1, 7[$ the set of upper bounds of B is $[7, +\infty[$, then, $\sup B = 7$.
The set of lower bounds of B is $] - \infty, 1]$, then, $\inf B = 1$.
3. $C =] - \infty, 5]$ the set of upper bounds of C is $[5, +\infty[$, then, $\sup C = 5$.
The set C is not bounded below, so $\inf C$ does not exist.

Supremum axiom

- Every nonempty set in $\mathbb{R}$ that is bounded above has a supremum.
- Every nonempty set in $\mathbb{R}$ that is bounded below has an infimum.

Proposition :

Let A and B be two nonempty sets in $\mathbb{R}$, then $\sup(A \cup B) = \max \left\{ \sup A, \sup B \right\}$ and $\inf(A \cup B) = \min \left\{ \inf A, \inf B \right\}$.

IV. Appendix 1 : Notions of logic

Before starting the course on real numbers, we will give some notions of mathematical logic.

IV.1 Glossary of mathematical statements

► **Proposition** : We call *proposition* a statement or expression that can be true or false.

Example 1.

- $3 + 5 = 0$. *Is false.*
- n, m two natural numbers, n is greater than m . (*This is not a proposition, it's a statement*)

► **Axiom** : We call *axiom* any proposition considered as evident, accepted as true without demonstration.

Example 2. *Euclide's Axiom asserts that through a given point A not on a line (D) , there is one and only one line in the plane of A parallel to (D) .*

► **Theorem** : We call *theorem* any proposition that is proven to be true.

► **Corollary** : A *corollary* is a direct consequence of a theorem.

► **Lemma** : We call *lemma* any true proposition preparatory to the establishment of a theorem of greater importance.

IV.2 Logical connectives

Logical connectives are used to define other propositions from one or more initial propositions. Let P and Q be two propositions, we define by :

► **Negation** : The *negation* of a proposition P is the proposition, denoted $\overline{P}$, which is true when P is false and false otherwise.

Example 3. $P : 3$ is an even number (**False**), $\overline{P} : 3$ is not an even number (**True**).

► **Conjunction** : The *conjunction* of two propositions P and Q is the proposition (P and Q), denoted $(P \wedge Q)$, which is true if P and Q are both true at the same time. It is false otherwise.

Example 4. *The proposition (2 divides 9) and (136 is a multiple of 17) is false, because the first one is false .*

► **Disjunction** : The *disjunction* of two propositions P and Q is the proposition (P or Q), denoted $(P \vee Q)$, which is true if at least one of the two propositions are true. It is false otherwise.

Example 5. *The proposition (2 divides 9) or (136 is a multiple of 17) is true, because the second is true.*

► **Implication (if...then)** : The proposition P implies Q , denoted $(P \implies Q)$ is true since if P is true, Q must be also true.

Remark. : The proposition $(P \implies Q)$ is also the same as $(\overline{P} \vee Q)$.

Example 6.

Let $x \in \mathbb{R}$.

1. $(x = 3) \implies (x^2 = 9)$ is true.
2. $(x^2 = 9) \implies (x = 3)$ is false.
3. $(x^2 = 9) \implies (x = -3 \text{ ou } x = 3)$ is true.
4. $(x > 3) \implies (x^2 > 9)$ is true.
5. $(x < -3) \implies (x^2 > 9)$ is true.

6. $(x < -3) \implies (x^2 < 9)$ is false.

Remark. From the implication $(P \implies Q)$, we define :

- The implication $(Q \implies P)$, called **converse** of the implication $(\implies Q)$.
- The implication $(\overline{Q} \implies \overline{P})$, called **contrapositive** of the implication $(\implies Q)$.
- The negation $\overline{(P \implies Q)}$ is the proposition $(P \wedge \overline{Q})$.

Example 7. Let be the following implication : $(n^2 \text{ is even }) \implies (n \text{ is even})$.

1. Its converse is : $(n \text{ is even }) \implies (n^2 \text{ is even})$.
2. Its contrapositive is : $(n \text{ is odd }) \implies (n^2 \text{ is odd})$.
3. Its negation is : $(n^2 \text{ is even })$ and $(n \text{ is odd})$.

► **Equivalence (if and only if) :** The proposition **P equivalent to Q** , denoted $(P \iff Q)$, is true since P and Q are both true.

In other words, it is the proposition $(P \implies Q) \wedge (Q \implies P)$.

Example 8. Let $x \in \mathbb{R}$. We have the following equivalence : $x^2 - 1 = 0 \iff x = -1, \text{ or } x = 1$. (the implication and its converse are both true).

Remark. In mathematical practice, we're only interested in true propositions. In other words, we'll write $P \iff Q$ or $P \implies Q$ only when they're true.

IV.3 Quantifiers

Let $P(x)$ the proposition dependant on the element x of a set E . We write

- $\forall x \in E, P(x)$: when the proposition P is true for all the elements $x \in E$.
 $\forall$, which can be read **whatever** or **for all**, is called **universal quantifier**.
- $\exists x \in E, P(x)$: when there exists at least an element x of the set E for which the proposition P is true.
 $\exists$, which can be read **there is at least one**, is called **existential quantifier**.
- $\exists! x \in E, P(x)$: when there exists a unique element x of the set E for which the proposition P is true.

There is jointly existence and uniqueness of the element x verifying the proposition P .

Example 9.

1. The proposition « $\forall n \in \mathbb{N}^*, n^2 - 1 \geq 0$ » is true.
2. The proposition « $\forall p \in \mathbb{N}, p^2 \geq 6$ » is false.
3. The proposition « $\exists n \in \mathbb{N}, n^2 - n > n$ » is true ($n=3, n=10, n=100$).
4. The proposition « $\exists x \in \mathbb{Z}, x^2 = -4$ » is false (no square of real number will give a negative number).
5. The proposition « $\exists! n \in \mathbb{N}, n^2 - n > n$ » is false.

IV.3.1 Negation of propositions depending on quantifiers

- The negation of $(\forall x \in E, P(x))$ is $(\exists x \in E, \overline{P(x)})$.

Example 10. The negation of $(\forall n \in \mathbb{N}^*, n^2 - 1 \geq 0)$ is $(\exists n \in \mathbb{N}^*, n^2 - 1 < 0)$.

- The negation of $(\exists x \in E, P(x))$ is $(\forall x \in E, \overline{P(x)})$.

Example 11. The negation of $(\exists n \geq 0, n^3 - n \text{ is a multiple of } 3)$ is $(\forall n \geq 0, n^3 - n \text{ it is not a multiple of } 3)$.

V. Appendix 2 : Some basic formulas in $\mathbb{R}$

Newton's binomial formula : For all $n \in \mathbb{N}$ and for all $a, b \in \mathbb{R}$, we have :

$$(a + b)^n = \sum_{k=0}^n C_n^k a^k b^{n-k}, \quad \text{with } C_n^k = \frac{n!}{k! (n-k)!}$$

The coefficients C_n^k are called binomial coefficients. To calculate these binomial coefficients, we can use the following Pascal's triangle

$n \backslash k$	k=0	k=1	k=2	k=3	k=4	k=5	k= 6 ...
n=0	1						
n=1	1	1					
n=2	1	2	1				
n=3	1	3	3	1			
n=4	1	4	6	4	1		
n=5	1	5	10	10	5	1	
n=6	1	6	15	20	15	6	1
$\vdots$							

TABLE 1 – Pascal's triangle

We obtain for example $C_5^2 = 10$, $C_4^3 = 4$, $C_6^3 = 20$, $C_6^2 = 15$, $C_3^1 = 3 \dots$

Remark : The binomial coefficient verify for all n in $\mathbb{N}$ the following properties :

$$C_n^n = 1, \quad C_n^0 = 1, \quad C_n^{n-1} = n$$

Example : Develop $(x + 2)^3$ and $(x - 1)^3$.

We have

$$(x + 2)^3 = \sum_{k=0}^3 C_3^k x^k 2^{3-k} = C_3^0 x^0 2^3 + C_3^1 x^1 2^2 + C_3^2 x^2 2^1 + C_3^3 x^3 2^0 = 8 + 12x + 6x^2 + x^3.$$

and

$$\begin{aligned} (x - 1)^3 &= \sum_{k=0}^3 C_3^k x^k (-1)^{3-k} = C_3^0 x^0 (-1)^3 + C_3^1 x^1 (-1)^2 + C_3^2 x^2 (-1)^1 + C_3^3 x^3 (-1)^0 \\ &= -1 + 3x - 3x^2 + x^3. \end{aligned}$$

Some of the first n integers : We have $\forall n \in \mathbb{N}, \quad 1 + 2 + 3 + \dots + n = \sum_{k=1}^n k = \frac{n(n+1)}{2}$.

Some of the n first terms of a geometric sequence : Let $x \in \mathbb{R} - \{1\}$, then

$$\sum_{k=0}^n x^k = 1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

$$\sum_{k=1}^n x^k = x + x^2 + \cdots + x^n = x \left(\frac{1 - x^n}{1 - x} \right) = \frac{x - x^{n+1}}{1 - x}.$$